A Vaccine Auction

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Abstract

This note describes an efficient way to auction off vaccines in a pandemic.

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1 Introduction

Suppose the government has bought forward vaccines from pharmaceutical companies. How should the government allocate these vaccines?

When triaging, doctors ask: How to allocate scarce resources to maximize the number of quality adjusted life years saved? When deciding how much pollution to allow, the Environmental Protection Agency asks: How to balance the number of lives saved (each valued at about ten million dollars) against the abatement costs? Imitating either approach in order to allocate vaccines in a pandemic would be unwise for two reasons:

1. No one apart from the individual himself knows how much he values the life-style of greater freedom afforded by the vaccine relative to the alternative of the social isolation. (Steven Landsburg makes this point here.[1])

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[1]Steven Landsburg, "Priority Care," *The Big Questions* blog, 2 December 2020.

2. There are externalities. Prioritizing the protection of the most vulnerable may do little to slow down the spread of the virus. (Tyler Cowen makes this point here.[2]) When infected, individuals impose externalities on their colleagues, employers, and their health insurer, private or public (such as the NHS in the United Kingdom).

A superior approach, described in this note, is an action with the following defining features:

- Each individual and each organization (e.g., a government agency, a charity, or a firm) can bid for a vaccine on behalf of any individual (including the bidder himself). While individuals (physical entities) get vaccinated, organizations do not.
- If getting vaccinated is more desirable earlier in the pandemic rather than later, then vaccines are allocated to the individuals in the descending order of the aggregate bids submitted on their behalf. That is, the individual with the greatest support as expressed by the sum of the bids submitted on his behalf gets the vaccine first; the individual with the second greatest support gets the vaccine second, and so on.
- Each bidder pays the externality that his bids impose on others by diverting the vaccine towards those on whose behalf he bids (including himself). Thus, the auction is not pay-as-bid. In particular, an individual with substantial bids submitted on his behalf may end up paying little or nothing for his own vaccine.

Because the described vaccine auction is a pivot VCG mechanism, it inherits the virtues shared by the auctions in the class:[3]

- The auction allocates the vaccine efficiently.
- The auction is strategy-proof: regardless of what others do, no bidder can do better than to bid his true valuations.

Here is how the auction mitigates the two major deficiencies of the triage-like vaccine distribution approach mentioned above:

[2] Tyler Cowen, "Vaccine Distribution Shouldn't Be Fair," *Bloomberg Opinion*, 23 November 2020.

[3] Ausubel and Milgrom (2005) discuss the virtues and the weaknesses of VCG (Vickrey–Clarke–Groves) mechanisms. The two bullet points below is their Theorem 1. Their Theorem 2 says that the described vaccine auction (and, in particular, its feature that bidders be allowed to bid on behalf of others) is essentially necessary under rather weak conditions provided one insists on the two bullet points.

1. The auction enables each bidder to express his private valuation for the vaccine. Examples:
   (a) A vaccine skeptic may express a negative valuation.
   (b) A nurse with virus-conferred immunity may express a zero valuation.
   (c) A youngster who looks after his ailing grandparents may express a large positive valuation.
2. The opportunity to bid on behalf of others enables individuals and organizations to internalize some of the externalities that vaccination entails. Examples:
   (a) A firm at the helm of a company town may subsidize the vaccination of (i.e., bid on behalf of) its employees in order to accelerate herd immunity and avert costly lockdowns.
   (b) A susceptible individual who is allergic to the vaccine may subsidize the vaccination of his doorman, his housekeeper, and his hairdresser.
   (c) A health insurance company (private or public) may subsidize the vaccination of the most vulnerable among the insured (e.g., the elderly, the obese, and those suffering from chronic conditions) in order to avoid paying their hospital bills.
   (d) An airline or a coffee shop chain may subsidize the vaccination of its loyal customers—and potential super-spreaders—in order to mitigate the treat they pose to other customers and to the company's reputation.

The described vaccine auction generates revenue. In order to preserve bidders' incentives, it is important that this revenue not be promised to any of the bidders. In particular, the budget of the public healthcare provider (e.g., the NHS), who is a bidder in the auction, may not be topped up depending on the auction's realized revenue. Instead, the government can commit to channeling the auction revenue towards retiring the national debt.

Related Literature

The problem of vaccine allocation is closely related to the thoroughly studied and successfully solved problem of selling ad positions to advertisers in online search. The ads are sold via auctions, popularized by *Google* and *Yahoo!*. The problem of selling online ads consists in allocating positions of sponsored links on a search page. The vaccine allocation problem consists in allocating positions in the vaccination queue. The former problem has been formalized by Edelman, Ostrovsky and Schwartz (2007) and Varian (2007), on whose formalization I build.

As far as allocating vaccines is concerned, the only difference from the problem of selling sponsored search positions is that a variety of actors might care about uncle Bob's vaccination, not just uncle Bob himself. The proposed vaccine auction accommodates this feature.

2 The Model

The model extends the positions auction environment of Edelman, Ostrovsky and Schwartz (2007) and Varian (2007) by permitting each bidder to bid not only on his behalf but also on behalf of others. The proposed auction is the standard pivot VCG mechanism.

Environment

In total I bidders participate in a vaccine auction. Typical bidders are indexed by i and j in $\mathcal{I} \equiv \{1, 2, \dots, I\}$. A bidder is interpreted as an individual or an organization.

Once a vaccine has been authorized for use, T units of the vaccine become available sequentially over time. (In practice, a "unit" comprises the first dose of the vaccine and any necessary boosters.) We say that the t-th unit becomes available at time $t \in \mathcal{T} \equiv \{1, 2, \dots, T\}$. Without loss of generality, let $T = I$. The (economic) time in the model need not correspond to the calendar time, as will be explained.

Bidders' common discount factor for getting vaccinated at time t is denoted by $\alpha_t \in \mathbb{R}_+$, which the auctioneer knows (more on which later). Each discount factor α_t is interpreted as the time-discounted reduction in the average mortality risk for those vaccinated at time t. Without loss of generality, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_T$. Any variation in the speed with which vaccine units become available is folded into the sequence $(\alpha_t)_{t \in \mathcal{T}}$; as a result, the calendar time will generally differ from the (economic) time in the model. For instance, a batch of k vaccine units that appear from time t onwards in short succession are all discounted similarly; $\alpha_t, \alpha_{t+1}, \dots, \alpha_{t+k-1}$ are all about the same. If the subsequent unit is shipped with much delay, then α_{t+k} is much lower than α_{t+k-1}, both because of impatience and because the pandemic continues to ravage. When herd immunity is reached, the sequence $(\alpha_t)_{t\in\mathcal{T}}$ falls precipitously. Moreover, if early vaccination is perceived as risky, then, all other things being equal, individuals may prefer to wait and see before getting vaccinated, in which case the convention $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_T$ would require that time $t = 2$ occur "before" time $t = 1$, where "before" is in terms of calendar time, while $t = 2$ and $t = 1$ refer to the (economic) time in the model.

Each bidder i values bidder j's vaccination at time t at $\alpha_t s_{ij}$, where $s_{ij} \in \mathbb{R}$. Here, s_{ii} pertains to bidder i's benefit from vaccinating himself and captures the bidder-specific adjustments to the mortality risk, the cost of mitigating exposure to the virus, and the value of life; and s_{ij} with $j \neq i$ pertains to the bidder's benefit from seeing someone else vaccinated (a relative, an employee, or an insuree). Each bidder i knows his valuations $\left(s_{ij}\right)_{j\in\mathcal{I}}$. A vaccine proponent has $s_{ii} \geq 0$, while a vaccine skeptic has $s_{ii} < 0$. A spiteful vaccine proponent has $s_{ij} < 0$ for some $j \neq i$.

Define an allocation $\mathbf{x} \equiv (x_{it})_{i\in\mathcal{I},t\in\mathcal{T}}$ by letting $x_{it} = 1$ if bidder i is vaccinated at time t and letting $x_{it} = 0$ otherwise. An allocation $\mathbf{x}$ is feasible if no two bidders are vaccinated at the same time: $\sum_{i\in\mathcal{I}} x_{it} \leq 1$ for all $t \in \mathcal{T}$. Let X denote the set of all feasible allocations.

At an allocation $\mathbf{x}$ in X, the payoff of a bidder i who makes a payment p_i is

$$\sum_{t\in\mathcal{T}} \sum_{j\in\mathcal{I}} x_{jt}\alpha_t s_{ij} - p_i.$$

Utility is transferable across bidders and the auctioneer, who collect $\sum_{i\in\mathcal{I}} p_i$.

The Vaccine Auction

Each bidder i submits a collection $\mathbf{b}_i \equiv \left(b_{ij}\right)_{j\in\mathcal{I}}$ of bids: a bid b_{ii} on his own behalf and a bid b_{ij} on behalf of each bidder $j \in \mathcal{I}\setminus\{i\}$. The aggregate bid on bidder i's behalf is denoted by $B_i \equiv \sum_{j\in\mathcal{I}} b_{ji}$.

The auction's allocation rule $\mathbf{x}^*$ is efficient; that is, it associates with each bid profile $\mathbf{b} \equiv (\mathbf{b}_i)_{i\in\mathcal{I}}$ a feasible allocation $\mathbf{x}^*(\mathbf{b})$ that maximizes the total surplus:

$$\mathbf{x}^*(\mathbf{b}) \in \arg\max_{\mathbf{x}\in X} \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{I}} x_{it}\alpha_t B_i. \tag{1}$$

Because the sequence $(\alpha_t)_{t\in\mathcal{I}}$ is weakly decreasing by assumption, one solves (1) by assigning each time-t unit of the vaccine to the bidder with the t-th highest aggregate bid, with indifferences resolved arbitrarily. That is, for any bidder i, we have $x^*_{it}(\mathbf{b}) = 1$ if and only if B_i is the t-th largest component of $\mathbf{B} \equiv (B_i)_{i\in\mathcal{I}}$.

Each bidder's payment in the auction is the externality that he imposes on other bidders assuming that all bidders bid truthfully, that is, assuming that $b_{ij} = s_{ij}$ for each i and j in $\mathcal{I}$. Formally, given the submitted bids $\mathbf{b} \equiv (\mathbf{b}_i, \mathbf{b}_{-i})$ with $\mathbf{b}_{-i} \equiv \left(\mathbf{b}_j\right)_{j\in\mathcal{I}\setminus\{i\}}$ and the induced aggregate bids $\mathbf{B}$, bidder i's payment is

$$p^*_i(\mathbf{b}) = \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}(0, \mathbf{b}_{-i})\,\alpha_t\left(B_j - b_{ij}\right) - \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}(\mathbf{b})\,\alpha_t\left(B_j - b_{ij}\right). \tag{2}$$

If all bidders bid truthfully, then the first term in (2) is the total surplus all but bidder i enjoy once the vaccines have been allocated efficiently while ignoring bidder i's bids, whereas the second term is the total surplus all but bidder i enjoy once the vaccines have been allocated without ignoring bidder i's bids. By construction, each payment (2) is nonnegative, and, therefore, the auction cannot possibly lose money.

An Equilibrium of the Vaccine Auction

Denote the payment function profile in the auction by $\mathbf{p}^* \equiv (p^*_i)_{i\in\mathcal{I}}$, where each function p^*_i is given in (2). Recall that the auction's allocation rule $\mathbf{x}^*$ is defined in (1). Proposition 1 shows that the game induced by the vaccine auction has a dominant-strategy equilibrium in which each bidder's every bid (on his own behalf and on behalf of others) equals his corresponding valuation.

Proposition 1. *The vaccine auction $(\mathbf{x}^*, \mathbf{p}^*)$ is efficient and strategy-proof: irrespective of what other bidders do, each bidder i cannot do better than bid truthfully by setting $b_{ij} = s_{ij}$ for each $j \in \mathcal{I}$. Moreover, if a bidder i bids truthfully and $s_{ij} \geq 0$ for all $j \in \mathcal{I}$, then his payoff is nonnegative.*

Proof. The proposition's conclusion lists standard properties of the VCG pivot mechanism, of which the described vaccine auction is an instantiation. A direct proof is in Appendix A. ■

One may intuitively wonder about "free-riding" on others' bids: Why would Alice ever bid her true valuation for getting vaccinated if she knows that her health insurer is prepared to bid tenfold on her behalf? The answer is that free-riding is built into the structure of the auction. Alice does not pay what she bids. She pays the externality that her bids impose. If her bids do not affect the allocation of the vaccine, and the bids of her heath insurer and others are doing all the work, then she pays nothing.

3 Some Practical Considerations

Discount Factors

The efficiency of vaccine allocation and the optimality of truthful bidding rely on the accuracy of the discount factors $(\alpha_t)_{t\in\mathcal{I}}$, which enter both (1) and (2). For efficiency, it suffices to know that the value of getting vaccinated decreases in calendar time, which is a reasonable a priori assumption. Indeed, a vaccine is likely to make the greatest difference as soon as it has been authorized for use, and less so as more and more individuals get vaccinated or acquire immunity through infection. In this case, it is efficient to vaccinate individuals in the descending order of the aggregate bids submitted on their behalf; the knowledge of the exact values of $(\alpha_t)_{t\in\mathcal{I}}$ is not required.

The exact values of $(\alpha_t)_{t\in\mathcal{I}}$ may not be available until after the pandemic. That's OK. The computation and the assessment of the payments can be delayed until after the pandemic without compromising truthful bidding at the beginning of the pandemic. In the context of positions auctions, Varian (2009) makes this point. Recall that each α_t is an estimate of the reduction in the average mortality risk for those who get vaccinated at time t. To determine $(\alpha_t)_{t\in\mathcal{I}}$ ex-ante, before the bids have been solicited, would require sophisticated epidemiological modeling and the forecasting of vaccine efficacy, infection mortality, and any medical treatments that could emerge on the way, as well as ample politico-economic guesswork. By contrast, $(\alpha_t)_{t\in\mathcal{I}}$ can be computed mechanically ex-post, once the evolution of the pandemic has been observed. In addition, the payments computed and assessed ex-post are "fair" in the sense that they charge bidders little for a vaccine that has proved inefficacious or has arrived too late to make a difference.

Vaccine Skeptics and Spiteful Individuals

The vaccine auction assumes that everyone participates. The auction guarantees a nonnegative payoff for unspiteful vaccine proponents (the "moreover" part of Proposition 1); such bidders are eager to participate. By contrast, vaccine skeptics and spiteful individuals may end up with negative payoffs.

To illustrate, suppose that there are two bidders (Alice and Bob) and two units of the vaccine with $\alpha_1 = \alpha_2 = 1$. Alice is a vaccine skeptic ($s_{11} = -1$) and does not care about Bob ($s_{12} = 0$). Bob seeks protection ($s_{22} = s_{21} = 2$). The auctioneer vaccinates both Alice and Bob, collects the payment of 1 from Bob and nothing from Alice. Alice's payoff is negative (-1). Even if Alice's objection to getting vaccinated exceeded Bob's benefit (i.e., if $s_{11} + s_{21} < 0$), her payoff would still be negative because she would pay the externality that her skipping the vaccine would impose on Bob. Alice's participation in the auction can be ensured either by coercion (vaccine passports would do the job) or by a subsidy (which may depend on Alice's observable characteristics but not on her bid).

A bidder's payoff can be negative even if no bidder is a skeptic as long as some bidder is spiteful. To illustrate, modify the example above by assuming that $s_{11} = 2$, $s_{12} = 0$, $s_{22} = 0$, and $s_{21} = -1$ (i.e., Bob is spiteful). The auctioneer vaccinates both Alice and Bob, collects the payment of 1 from Alice and nothing from Bob. Bob's payoff is negative (-1). Even if Bob's objection to Alice's vaccination exceeded her benefit (i.e., if $s_{21} + s_{11} < 0$), his payoff would still be negative because he would pay the externality that his blocking of Alice's vaccination would impose on her. Bob's negative payoff would not discourage him from participating in the auction, for nonparticipation does not mute his hurt when others receive the vaccine.

Both examples illustrate that the auction does not discriminate between "moral" and "immoral" preferences, and takes full advantage of the model's assumption that interpersonal grievances are comparable and transferable.

Just as vaccine skepticism can have an "innocent" explanation (an allergy), so can spitefulness: a hospital may have a financial interests in its patients' contracting the infection thanks to a reimbursement from the insurer. To illustrate, suppose that there are three bidders (Alice, Hospital, and Insurer) and one unit of the vaccine, with $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$. Alice is a vaccine proponent ($s_{11} = 1$). Hospital profits from treating unvaccinated Alice ($s_{21} = -2$). Insurer must reimburse Hospital and experiences some overhead ($s_{31} = 3$). The remaining valuations are zero. In equilibrium, Alice is vaccinated because $s_{11} + s_{21} + s_{31} > 0$. Alice and Hospital pay nothing, and Insurer pays 1. The example illustrates that the vaccine auction is not deceived by the contractual transfers between the bidders (here, the transfer from Insurer to Hospital, implicit in their valuations) and identifies the efficient allocation correctly.

"Lazy" (or Inattentive) Bidding

One may be concerned about bidders being insufficiently motivated to discover how much they value the vaccine, especially when it is administered to others. Could a bidders could submit a "lazy"—inaccurate—bid (e.g., a zero) at no cost to himself but to the great detriment to efficiency? No. The concern is unjustified. In the vaccine auction, each bidder's payoff is the total surplus plus a constant. A "lazy" bid that affects the total surplus has payoff consequences.

Moreover, the result of Bergemann and Välimäki (2002, Corollary 1) applies: the vaccine auction provides bidders with socially optimal incentives to discover their valuations. That is, in the model's extension in which each bidder chooses the precision of a costly signal about his valuations, under appropriate separability assumptions on the costs, the vaccine auction induces a game that has an equilibrium in which each bidder acquires the amount of information that maximizes the expected total surplus generated by the auction. In other words, any bidder inattention to valuations is not only privately rational but is also socially optimal (i.e., efficient).[4]

Self-Image and Privacy

The described vaccine auction forces each bidder to confront the potentially uncomfortable problem of assigning a cash value to his own life, as well as to the lives of relatives, co-workers, and employees. "Am I the kind of person capable of, first, cold-heartedly putting a price tag on a life and then living with the memory of this number for the rest of my life? Would I like others to know whose lives I value and how much?" If the answer to either question is no, then a bidder may refrain from participating in the auction, in order to protect his self-image or privacy, or both. Following Benabou and Tirole (2006), self-image can be modeled as the inference of one's own valuations, suppressed or forgotten once the auction is over, from the allocation and the payments selected by the auction. The loss of privacy refers to the same inference performed by others. A so-called "differentially private" modification of the vaccine auction mitigates both concerns.

[4] Bergemann and Välimäki's (2002) result also implies that the vaccine auction motivates the bidders to undertake socially optimal investments in enhancing their valuations of the vaccine.

Roughly speaking, a differentially private auction ensures that no single bidder can significantly affect the probability distribution over the auction's outcomes, regardless of how others bid. Because the outcome of a differentially private auction depends little on the bids of any single bidder, it is impossible to infer any bidder's bids with much precision by inverting the auction's outcome. The vaccine auction described in this note is not differentially private; a bidder can either deny or guarantee himself an early vaccination by submitting appropriate bids. A differentially private modification of the vaccine auction can be obtained by carefully injecting randomness into the auction's allocation and payments. Huang and Kannan (2012) show how this can be done for any VCG mechanism. The implied modified vaccine auction admits a little inefficiency in exchange for some differential privacy while remaining strategy-proof.

4 Concluding Remarks

1. One criticism that one can level against the vaccine auction is that it neglects the fact that some people are poor, others are rich, and the rich will be able to pay more for a vaccine than the poor, which is somehow unfair. Another way to say this is that the rich have lower "marginal utility for money" than the poor do. Dworczak, Kominers and Akbarpour (2020) point out this problem and propose a solution, which involves rationing and price controls. This approach is not pursued in this note. Instead, I observe that:

   (a) The proposed vaccine auction takes care of the poor to the extent that it motivates the healthcare provider (private or public) to bid generously on behalf of the poor as long as the poor are also sick, possibly elderly, and with chronic conditions, all of which make contracting the virus extremely costly for the healthcare provider. As a result, beneficiaries of generous bids by the healthcare provider, the poor may end up paying little or nothing for their vaccine.

   (b) In the cases in which the vaccine auction does not motivate others to to bid generously on behalf of the poor, it is questionable whether it is morally imperative—or indeed defensible—to use a vaccine allocation mechanism to rectify pre-existing inequalities. Perhaps, these inequalities reveal the social preference that ought to be respected. Or perhaps, while in conflict with the social preference, pre-existing inequalities can be rectified with greater legitimacy by slow democratic deliberation once the pandemic has been conquered rather than by emergency fiat in the midst of a pandemic. If the emergency fiat it must be, then it may be best (if possible) to separate efficiency from redistribution; if an identifiable group is deemed to be undeservedly poor, it is best to give them cash, which they can then spend on vaccines or on something else, rather than force them to consume vaccines.

2. A superficially compelling alternative to the vaccine auction is, first, to allocate the vaccines however the government sees fit (or even to allocate randomly) and then to allow individuals to trade their vaccination priorities among themselves. The problem with this approach is the ineluctable inefficiency that arises from the Myerson and Satterthwaite (1983) impossibility theorem. Once private ownership is introduced, markets need not put vaccines into the hands of the individuals with the highest valuations when the potential buyers' and the potential sellers' valuations are their private information.

A Proof of Proposition 1

Efficiency is by construction of the allocation rule $\mathbf{x}^*$ in (1).

For strategy-proofness, note that, for any collection $\mathbf{b}_{-i}$ of others' bids, for any collection $\mathbf{b}_i$ of bidder i's bids, and for any collection $\mathbf{s}_i \equiv \left(s_{ij}\right)_{j\in\mathcal{I}}$ of bidder i's true valuations, the payoff from bidding the $\mathbf{s}_i$ weakly exceeds the payoff from bidding $\mathbf{b}_i$:

$$
\begin{aligned}
&\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}\left(\mathbf{s}_i,\mathbf{b}_{-i}\right)\alpha_t s_{ij} - p^*_i\left(\mathbf{s}_i,\mathbf{b}_{-i}\right)\\
&\qquad = \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}\left(\mathbf{s}_i,\mathbf{b}_{-i}\right)\alpha_t\left(\sum_{j'\in\mathcal{I}\setminus\{i\}} b_{j'j} + s_{ij}\right) - \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}\left(0,\mathbf{b}_{-i}\right)\alpha_t \sum_{j'\in\mathcal{I}\setminus\{i\}} b_{j'j}\\
&\qquad = \max_{\mathbf{x}} \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x_{jt}\alpha_t\left(\sum_{j'\in\mathcal{I}\setminus\{i\}} b_{j'j} + s_{ij}\right) - \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}\left(0,\mathbf{b}_{-i}\right)\alpha_t \sum_{j'\in\mathcal{I}\setminus\{i\}} b_{j'j}\\
&\qquad \geq \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}\left(\mathbf{b}_i,\mathbf{b}_{-i}\right)\alpha_t\left(\sum_{j'\in\mathcal{I}\setminus\{i\}} b_{j'j} + s_{ij}\right) - \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}\left(0,\mathbf{b}_{-i}\right)\alpha_t \sum_{j'\in\mathcal{I}\setminus\{i\}} b_{j'j}\\
&\qquad\qquad = \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}\left(\mathbf{b}_i,\mathbf{b}_{-i}\right)\alpha_t s_{ij} - p^*_i\left(\mathbf{b}_i,\mathbf{b}_{-i}\right),
\end{aligned}
$$

where the first and the last equalities use (2), the second equality uses (1), and the inequality replaces an allocation rule that is optimal for $(\mathbf{s}_i, \mathbf{b}_{-i})$ by a rule that need not be optimal for $(\mathbf{s}_i, \mathbf{b}_{-i})$ (but happens to be optimal for $(\mathbf{b}_i, \mathbf{b}_{-i})$). Strategy-proofness follows by combining the chain of equalities and the inequality.

For the "moreover" part, the first line in the display above can be rewritten using (1) as

$$\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x^*_{jt}\left(\mathbf{s}_i, \mathbf{b}_{-i}\right)\alpha_t s_{ij} - p^*_i\left(\mathbf{s}_i, \mathbf{b}_{-i}\right) = \max_{\mathbf{x}\in X}\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x_{jt}\alpha_t\left(\sum_{j'\in\mathcal{I}\setminus\{i\}} b_{j'j} + s_{ij}\right) - \max_{\mathbf{x}\in X}\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{I}} x_{jt}\alpha_t\left(\sum_{j'\in\mathcal{I}\setminus\{i\}} b_{j'j}\right),$$

which is nonnegative when $s_{ij} \geq 0$ for each i and j in $\mathcal{I}$.

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